THE CONNECTION OF A RICKART REAL C*-ALGEBRA WITH ITS ENVELOPING RICKART (COMPLEX) C*-ALGEBRA

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Abstract: In the paper Rickart complex and real C*-algebra are consider. For Rickart real C*-algebra, its connection with the enveloping (complex) C*-algebra is studied. It is shown that the fact that A is a Rickart real C*-algebra does not imply that a complexification A +i A of A is a Rickart (complex) C*-algebra. Proved that if A is a real C*-algebra and A +i A is a Rickart C*-algebra, then A is a Rickart real C*-algebra. It is shown that there exists a Rickart real C*-algebra whose projection lattice is not complete.

Keywords: C*-algebra; Rickart complex and real C*-algebras; complex and real AW*-algebras.

Introduction: The theory of operator algebras was initiated in a series of papers by Murray and von Neumann in thirties. Later such algebras were called von Neumann algebras or W*-algebras. These algebras are self-adjoint unital subalgebras M of the algebra B(H) of bounded linear operators on a complex Hilbert space H, which are closed in the weak operator topology. Equivalently M is a von Neumann algebra in B(H) if it is equal to the commutant of its commutant (von Neumann's bicommutant theorem). A factor (or W*-factor) is a von Neumann algebra with trivial centre and investigation of general W*-algebras can be reduced to the case of W*-factors, which are classified into types I, II and III.

Rings and algebras, which will be discussed below, first studied by C.E. Rickart [1]. These algebras were further developed in the works of I. Kaplansky [2-4]. Exactly, AW*-algebras were proposed by I. Kaplansky as an appropriate setting for certain parts of the algebraic theory of von Neumann algebras. In this article, we consider the real analogue of these algebras.

Preliminaries:

Definition 2.1. A C*-algebra is a (complex) Banach *-algebra whose norm satisfies the identity $\|x^*x\|=\|x\|^2$. Now let A be a real Banach *-algebra. A is called a real C*-algebra, if $A_c = A + iA$ can be normed to become a (complex) C*-algebra by extending the original norm on A.

Note that a C*-norm on A_c is unique, if it exists. It is known that [5, Corollary 5.2.11]

A is real C*-algebra if and only if $\|x^*x\|=\|x\|^2$ and $1+x^*x$ is invertible, for any $x \in A$.

Definition 2.2. If A is a ring and S is a nonempty subset of A, we write

$R(S)=\{x\in A : sx=0, \quad \forall s\in S\}$

and call R(S) the right-annihilator of S. Similarly,

$L(S)=\{x\in A : xs=0, \quad \forall s\in S\}$

denotes the left-annihilator of S.

Definition 2.3. A Rickart *-ring is a *-ring A such that, for each $x \in A$, $R(\{x\}) = gA$ with g a projection (note that such a projection is unique). It follows that $L(\{x\}) = (R(\{x^*\}))^* = (hA)^* = Ah$ for a suitable projection h. A (complex) C*-algebra that is a Rickart *-ring will be called a Rickart C*-algebra

The connection of a rickart real c*-algebra with its enveloping rickart (complex) c*-algebra: In this section, we consider the connection of a Rickart real C*-algebra with its enveloping Rickart (complex) C*-algebra. There is example of a Rickart real C*-algebra for which the enveloping C*-algebra (i.e., its complexification) is not a Rickart (complex) C*-algebra.

By applying the scheme of proof of [7, Proposition 4.2.3] we obtain:

Theorem 3.1. The complex C*-algebra B+iB is not a Rickart C*-algebra.

Now let us consider the converse problem: if A is a real C*-algebra and A+iA is a Rickart C*-algebra is A necessarily a Rickart real C*-algebra?

The following result gives a positive answer to this problem, which is the main result of this section.

Theorem 3.2. Let A be a real C*-algebra and let $A_c = A + iA$ be its complexification. Suppose that A_c is a Rickart C*-algebra. Then A is a Rickart real C*-algebra.

Main results. In this section, we will show that there is a Rickart real C*-algebra whose projection lattice is not complete. We will consider the connection between Rickart real C*-algebra and real AW*-algebra.

Proposition 4.1. [6, Proposition 3, paragraph 3]. Let A be a Rickart *-ring and $x \in A$. There exists a unique projection e such that (1) $xe = x$, and (2) $xy = 0$ iff $ey = 0$. Similarly, there exists a unique projection f such that (3) $fx = x$, and (4) $yx = 0$ iff $yf = 0$. Explicitly, $R(\{x\}) = (1-e)A$ and $L(\{x\}) = A(1-f)$. The projections e and f are minimal in the properties (1) and (3), respectively. We write $e = RP(x)$, $f = LP(x)$, called the right projection and the left projection of x.

It is known that any C*-algebra can be isomorphically embedded into some $B(H)$. Namely, there is an isomorphism of a C*-algebra onto a uniformly closed C*-subalgebra in $B(H)$, for some complex Hilbert space H. On the other hand, if $A \subset B(H)$ be a real C*-algebra, then it is also known that there is a real Hilbert space H_r with

$$H_r + iH_r = H, \quad A \subset B(H_r) \subset B(H_r) + iB(H_r) = B(H).$$

Proposition 4.2. The real C*-algebra $B(H_r)$ is a Rickart real C*-algebra. Explicitly, if $x \in B(H_r)$ then $LP(x)$ is the projection on the closure of the range of x, and $1 - RP(x)$ is the projection on the null space of x (i.e. $1 - RP(x) \in x^*(H_r)^{\perp}$).

If A is a Rickart *-ring and B is a *-subring of A, then B need not be a Rickart *-ring: an obvious example is when B has no unity element (it's obvious that a Rickart *-ring always has a unity element), but adjoining a unity element may not be a remedy (we will show it below). There is, nevertheless, a useful positive result.

Proposition 4.3. [6, Proposition 8, paragraph 3]. Let A he a Rickart *-ring and let B be a *-subring such that (1) B has a unity element, and (2) $x \in B$ implies $RP(x) \in B$. Then B is also a Rickart *-ring.

Theorem 4.1. There exists a Rickart real C*-algebra whose projection lattice is not complete.

Definition 4.1. A Bear *-ring is a *-ring A such that, for every nonempty subset $S \subset A$, $R(S) = gA$ for a suitable projection g. It follows that $L(S) = R(S^*)^* = (hA)^* = Ah$ for a suitable projection h.

The relation between Rickart *-rings and Baer *-rings is the relation between lattices and complete lattices:

Proposition 4.4. [6, Proposition 1, paragraph 4]. The following conditions on a *-ring A are equivalent:

(a) A is a Baer *-ring;

(b) A is a Rickart *-ring whose projections form a complete lattice;

(c) A is a Rickart *-ring in which every orthogonal family of projections has a supremum.

Definition 4.2. A (complex) C*-algebra that is a Bear *-ring will be called a AW*-algebra. A real C*-algebra that is a Bear *-ring will be called a real AW*-algebra.

By Theorem 4.1 and Proposition 4.4 we obtain

Corollary 4.1. A real AW*-algebra is a Rickart real C*-algebra, but the converse is not true, i.e., a Rickart real C*-algebra doesn't need to be a real AW*-algebra.

Corollary 4.2. The algebra B from the Example above is a Rickart real C*-algebra, but by Theorem 3.1 it is not a real W*-algebra.

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